Bounds on the Lifetime of Wireless Sensor Networks

Juan Alonso, Adam Dunkels, Thiemo Voigt
Swedish Institute of Computer Science

November 16, 2004

:

SICS Technical Report T2004:13
ISSN 1100-3154
ISRN:SICS-T–2004/13-SE
Keywords: sensor networks, lifetime, bounds

Abstract

Energy is one of the most important resources in wireless sensor networks. We use an idealized mathematical model to study the energy consumption under all possible routings. Our results are very general and, within the assumptions listed in Section 2, apply to arbitrary topologies, routings and radio energy models. We find bounds on the minimal and maximal energy routings will consume, and use them to bound the lifetime of the network. The bounds are sharp, and we show that they are achievable in many situations of interest. We give some examples, and apply the theory to the problem of covering a given square region with the most efficient member of a family of increasingly more dense square-lattice sensor networks. Finally, we use simulations to test these results in a more realistic scenario, where packet loss can occur.

1 Introduction

Recent technological advances have made the production of small and inexpensive wireless sensor devices possible, prompting a flurry of research and experiment. The starting point for this paper was a statement by Mainwaring et al. [4], one of the initial exciting deployments of this type of sensors. When discussing different routing algorithms, the authors write (in Section 6.2): Although these methods provide factors of 2 to 3 times longer network operation, our application requires a factor of 100 times longer network operation... . We thought this was intriguing: What factor is reasonable to expect of a routing algorithm? Typically, communication is the most expensive activity in terms of energy [5].

In this paper we focus on the energy consumed in communication, regardless of the particular routing scheme used. We consider only the energy required to receive and send data. We address questions such as: How much improvement in the lifespan of a network can be expected by changing only the routing algorithm? Which factors, as far as routing is concerned, affect the networks lifespan the most? How good is my favorite routing?

There is a vast literature relating energy consumption to routing, see for instance [2, 8, 6, 3] and the references in these papers. With few exceptions (see e.g. [1]) this previous work has concentrated on the performance of specific routing algorithms. Our main contribution is to provide fundamental limits to the energy consumption of routings, applicable regardless of the topology, routing algorithm, or radio energy model.

As an application, we study the problem of covering a square region of fixed size with a family of increasingly more dense square-lattice sensor networks. Using a specific radio energy model in our abstract results, we compute the essentially unique member of the family that consumes least energy. Finally, we use simulation to test these results in a more realistic scenario, where packet loss can occur.

The paper is organized as follows. Section 2 summarizes the assumptions we make on the sensor networks. Routings and the way we measure energy consumption are defined in Section 3. Section 4 contains Theorem 1, the main theoretical result of the paper, giving lower bounds on the energy consumption of routings. These results are applied in Section 5 to bound the lifetime of a sensor network. In Section 6 we give examples to illustrate the theory developed in the previous sections, and in Section 7 discuss whether the lower bounds of Theorem 1 can be achieved. Section 8 contains the above mentioned application to square-lattice sensor networks, and the simulation results are described in Section 9. The final Section 10 presents conclusions and future work.

2 Assumptions on the sensor network

We assume the nodes in the network are of two types: sensor nodes and base nodes. Sensor nodes (or, simply, nodes) are low-energy and have very limited memory and processing capabilities, whereas base nodes are high-energy and have significantly more processing power and memory capacity than sensor nodes. We make the assumption that there is an underlying hierarchic architecture whereby the base nodes control the sensor nodes deciding, in particular, which routing to use.We use the term routing to denote a specific set of paths (or multi-paths) that packets take through a network. A routing is the result of the particular routing algorithm used. The sensor nodes take readings and send them to the bases using other sensor nodes to reach them. This process is repeated until nodes die, eventually breaking connectivity and making the network non-operational. Another assumption is that during the whole process all nodes transmit at the same, constant power. No data aggregation is done in the network: all data gathered is sent unchanged to the base nodes.

3 Routings and their energy consumption

We model the network by a directed graph $G = (V, E)$. Given a link $e = (v, w)$, we let $\overline{e} = (w, v)$ denote the reverse link. We assume that if $e \in E$ then also $\overline{e} \in E$. We assume given a set $B \subseteq V$ of base nodes with $0 < |B| < |V|$.

The network operates with the following traffic pattern. For each iteration $t, 1 \leq t \leq T$, every node sends a packet of a certain length to some base node. Informally, each way to do this is a routing. More formally, a *routing* is a vector

$$y = (y_e^t)_{1 \leq t \leq T, e \in E}$$

where y_e^t represents the total number of packets destined to some base node that are sent through e during the t:th iteration. Observe that we can think of the routing y as being a sequence $y = (y^1, ..., y^T)$, where y^t is the routing used during the t:th iteration. The only restriction we place on routings is that they should be *effective*, in the sense of not having loops. A *routing has no loops* if for all $1 \leq t \leq T$ the following holds: for every node, the directed path used to send its packet to a base node never visits a node more than once. Let

$$R^T = \{y = (y_e^t)_{1 \leq t \leq T, e \in E} | y \text{is a routing with no loops}\}$$

The energy consumption of a routing y will be measured by the following cost function $f^T : R^T \rightarrow \mathbf{R}_+$:

$$f^T(y) = max_{v \in V} \{\sum_{t=1}^{T} (\sum_{e \in I^v} \rho y_e^t + \sum_{e \in O^v} \tau y_e^t)\} \tag{1}$$

where ρ [resp. τ] is the cost for the reception [resp. transmission] of one packet, I^v is the set of incoming links of $v, I^v = \{(i,j) \in E | j = v\}$, and O^v is the set of outgoing links of $v, O^v = \{(i,j) \in E | i = v\}$. Thus, $f^T(y)$ measures the maximum energy used by nodes when transmitting and receiving according to routing y . When $T = 1$ we write simply $f(y)$.

4 How parsimonious is my favorite routing?

Set $m^T = min_{f \in R^T} f^T(y)$, and $M^T = max_{f \in R^T} f^T(y)$. We thus have, for an arbitrary routing $y \in R^T, m^T \leq f^T(y) \leq M^T$. When $T = 1$ we write simply m, M. In this section we find bounds on the size of the interval $[m^T, M^T]$. For this purpose, we partition the set of nodes into subsets $S_0, ..., S_n$ satisfying $V = S_0 \cup S_1 ... S_n$, $S_i \cap S_j = \emptyset$ for all $i = j$, and no S_i is empty. The definition of the S_i is as follows: $S_0 = B$, and for $i > 0$, S_i is the set of nodes can be reached in i hops, but not less than i hops, from some node in S_0 (i.e. S_i is the "sphere" of radius i around S_0). Thus,

$$|V| = |S_0| + |S_1| + ... + |S_n|$$

and all $|S_i| > 0$. Notice that $n \geq 1$, since $|B| < |V|$. Corresponding to the spheres S_i , there are "balls" of radius i, denoted B_i, and defined by $B_i = S_0 \cup ... \cup S_i$. It will be convenient to introduce the following notation: $s_i = |S_i|$, $b_i = |B_i|$, and $N = |V|$. Finally, for $i = 1, ..., n$, we set:

$$m_i = \frac{N - b_i}{s_i} * \rho + \frac{N - b_i + s_i}{s_i} * \tau \tag{2}$$

These constants are interesting because of Theorem 1 below. Inequalities (i)-(iii) of the theorem can be seen as providing fundamental limits to the possible amount of improvement in energy consumption that can be derived from changes in the routing algorithm, and benchmarks to compare your favorite routing(s) against. The strength of Theorem 1 derives from its generality, as its results apply to any graph, routing, and radio energy model.

Theorem 1. *With the notation above,*
i) $M^T \leq T * [\rho * (N - s_0 - 1) + \tau * (N - s_0)] = T[(\rho + \tau) * (N - s_0) - \rho]$
ii) $m^T \geq T * max\{m_1, ..., m_n\}$

iii) $M^T \leq s_1 * m^T + T * \rho(s_1 - 1)$
iv) $m_n = \tau$.

Proof. Notice first that (iv) follows immediately from the definitions, since $N = b_n$. Next, for arbitrary $v \in V$and $y \in R^T$, notice that $\sum_{e \in I^v} y_e$ is the total number of packets received by v and, likewise, $\sum_{e \in O^v} y_e$ is the total number of packets transmitted by v. We claim these numbers cannot exceed the total number of packets being sent throughout the network at each iteration, i.e. $N - s_0$ packets transmitted and $N - s_0 - 1$ packets received. This is true because y has no loops and hence v will receive and send at most one packet for every non-base node. (i) follows immediately from this.

To prove (ii) it suffices to prove that $m^T \geq T * m_i$, for all $1 \leq i \leq n$. The idea of the proof is to consider S_i as a bottleneck for nodes outside B_i trying to reach S_0. More formally, notice that in every routing, packets in $V \backslash B_{i-1}$ can only reach S_0 by either going through S_i (i.e. these packets originate outside of B_i and, hence, are both received and transmitted by some element of S_i) or by being transmitted by some node in S_i (i.e. these packets originate at S_i). Thus, the nodes in S_i must receive $N - b_i$ packets, and they must transmit $N - b_i + s_i$ packets . For every $y \in R$ we have:

$$\begin{aligned} f^T(y) &\geq max_{v \in S_i} \{ \sum_{t=1}^{T} (\sum_{e \in I^c} \rho * y_e^t + \sum_{e \in O^v} \tau * y_e^t) \} \\ &\geq \rho * T * \frac{N - b_i}{s_i} + \tau * T * \frac{N - b_i + s_i}{s_i} = T * m_i \end{aligned} \tag{3}$$

Inequality (3) follows from Lemma 1 below. Indeed, by the discussion above, $\sum_{v \in S_i} (\sum_t \sum_{e \in I^v} \rho * y_e^t) = T * \rho * (N - b_i)$, and $\sum_{v \in S_i} (\sum_t \sum_{e \in O^v} \tau * y_e^t) = T * \tau * (N - b_i + s_i)$. Applying Lemma 1 to the sum of these two sums yields (3), as desired. Next, since $f^T(y) \geq T * m_i$ holds for all $y \in R$, we obtain $m^T \geq T * m_i$, as desired. Finally, using (i) and $T * m_1 \leq m^T$, we get

$$\begin{aligned} M^T &\leq T * \rho * (N - s_0 - 1) + T * \tau * (N - s_0) \\ &= T * \rho (N - b_1) + T * \rho * (s_1 - 1) + T * \tau * (N - s_0) \\ &= T * m_1 * s_1 + T * \rho * (s_1 - 1) \end{aligned} \tag{4}$$

This completes the proof of (iii) and of the theorem.

Lemma 1. *Let I denote a finite set. If* $\sum_{i \in I} A_i \geq \frac{a}{|I|}$*, then*

$$max\{A_i | i \in I\} \geq \frac{a}{|I|}$$

Proof. Suppose, for contradiction, that the conclusion of the Lemma is false. Then $A_i < a/|I|$, for all $i \in I$. But then $\sum_I A_i < \sum_I a/|I| = a$, contradicting the hypothesis. This proves the lemma.

It is meaningful to distinguish two cases in Theorem 1, according to whether or not $n = 1$. We consider first the rather trivial case when $n = 1$, i.e. when all nodes are one hop away from a base node. Inequality (ii) in Theorem 1 reduces to $m^T \geq T * \tau$, i.e. the minimal energy use after T iterations is the transmission cost times T . It is easy to find an optimal routing, i.e. a routing achieving this minimum: for each node, select a unique base node one hop away, and transmit the node's unique packet to the chosen base node; repeat T times. In this case, the upper bound for $M^T, T * [(\rho + \tau) * s_1 - \rho]$, can be achieved if, for instance, the non-base nodes can use each other to transmit their packets to a specified non-base node that receives all the packets minus its own, and transmits all s_1 packets to a base node. Summarizing:

Corollary 1. *In the special case when every node is only one hop away from a base node, we have:*
i) $M^T \leq T * [(\rho + \tau) * s_1 - \rho]$
ii) $m^T \geq T * \tau$
Moreover, (ii) is a sharp bound, i.e. there is a routing $y \in R^T$ with $f^T(y) = m^T$.

One can obtain a nicer form for the coefficient in Theorem 1(iii), by moving from the above case, when S_0 is "thick", to the opposite case, when S_1 is "thin" in the sense that $s1 \leq s_2 + ... + s_n$ or, equivalently, when $N - b_1 \geq s_1$. In this case Theorem 1 takes the neater form expressed in Corollary 2. The corollary says that, in terms of f^T -value, no routing is worse than $2 * s_1 - 1$ times the best possible routing. This gives an answer to a question asked in Section 1: What factor is reasonable to expect of a routing algorithm?

Corollary 2. *Suppose the network contains many nodes at least two hops away from al l base nodes, i.e. that $N - b_1 \geq s_1$. Then for all $T \geq 1$:*

$$M^T \leq (2s_1 - 1) * m^T$$

Proof. The condition $N - b_1 \geq s_1$ implies that m_1 in (2) satisfies $m_1 \geq \rho$. Together with equation (4) this gives $M^T \leq T * m_1 * (2s_1 - 1) \leq (2s_1 - 1) * m^T$, since $T * m_1 \leq m^T$ by Theorem 1(ii). This completes the proof.

5 Bounds on the lifetime of a sensor network

Suppose each node has the exact same amount EE of energy and we use a routing y in a traffic pattern consisting of T iterations. The network will be operational as long as $f^T(y) \leq EE$ and, to compute the break point[1], we set $f^T(y) = EE$, and let T_{max} denote the corresponding value of T . The next theorem bounds the life of the network in terms of T_{max}.

Theorem 2. *The maximum number T_{max} of readings a sensor network can take under the given assumptions is bounded as follows*:

$$\frac{EE}{(\rho + \tau) * (N - s_0) - \rho} \leq T_{max} \leq \frac{EE}{max\{m_1, ..., m_n\}}$$

Proof. It follows from $f^{T_{max}}(y) = EE$, that $m^{T_{max}} \leq EE \leq M^{T_{max}}$. Applying Theorem 1 to these inequalities, $T_{max} * max\{m_1, ..., m_n\} \leq EE \leq T_{max} * [(\rho + \tau) * (N - s_0) - \rho]$. Theorem 2 follows immediately from this.

6 Examples

The results of Sections 4 and 5 highlight the role of the spheres S_i in the longevity of a sensor network. Notice that the m_i (see equation (2)) decrease as s_i increases, suggest-

[1] T_{max} is time to first node failure. When (iii) of Thm.1 is sharp, i.e. when, say, $m^T = T * max\{m_1, ..., m_n\} = T * m_i$, all nodes of the sphere S_i will fail at the same time, breaking connectivity. That $m^T > T * max\{m_1, ..., m_n\}$ indicates that it is not possible to balance traffic evenly. This lends support to the conjecture that the portion(s) of the network depending on the corresponding dead node(s) to reach B , will be disconnected.

ing that the larger the s_i are, the more one stands to gain by devising and implementing smart routing algorithms, i.e. those that exactly, or nearly so, achieve the minimum value m^T . This brings us to the question as to how sharp is bound (ii) of Theorem 1, which is also related to the question of whether increasing the s_i will always result in an increased lifespan for the network. Ways to increase s_i are, e.g. to place more sensor nodes in the vicinity of the given ones, and/or to increase the transmission range. Theorems 1 and 2 show that $max\{m_1, ..., m_n\}$ is the theoretical minimum for the energy consumption of routings. Examples 1 and 5 below illustrate two different ways in which the theoretical value can fail to be achieved, i.e. $m \neq max\{m_i\}$. In both cases it is impossible to balance the load evenly among the nodes of S_1 . Example 2 shows that $max\{m_i\}$ need not equal m_1 . Example 3 is a simple case to illustrate that using the same routing at each iteration can be far from optimal. Example 4 is characteristic for rectangular networks with ”judicious” choice of base nodes, while Example 5 shows that size and placement of the base are important parameters in order to obtain the most of the network.

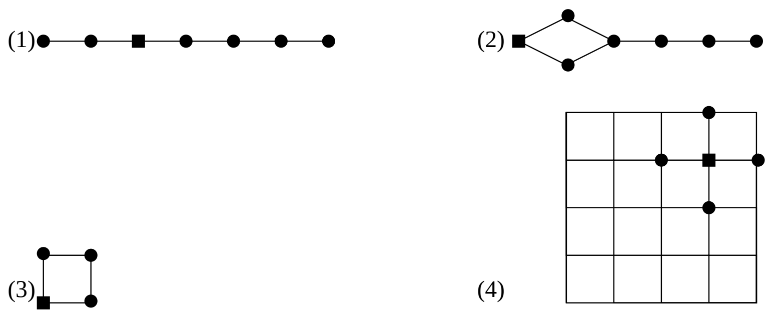

Figure 1: Networks (the sqaure node is the base)

1. Consider network (1) of Fig.1, where B consists of the square node. The network consists of two trees rooted at the base node. In this case $m_1 = 2\rho + 3\tau$, $m_2 = m_3 = \rho + 2\tau$, $m4 = \tau$, and $max\{m_i\} = m_1$. However, $m = f(y) = 3\rho + 4\tau > m_1$, where y is the only routing without loops on each of the rooted trees.

2. The network in Fig.1(2) illustrates a case where $max\{m_i\} \neq m_1$. In this case, according to equation (2), $m_1 = 2\rho + 3\tau = m_3$, $m_2 = 3\rho + 4\tau$, $m_4 = \rho + 2\tau$, and $m_5 = \tau$. It is easy to see that $m = m_2 = max\{m_i\}$.

3. For the graph in Fig.1(3), $m_1 = (1/2)\rho + (3/2)\tau$, and $m_2 = \tau$. In this case $m_1 = max\{m_i\}$, but $m^T \neq T * m_1$. However, if we let $y = (y_1, y_2, y_1, y_2, ...)$, then $f^T(y) = m^T$ ”on average”, in the sense that $f^T(y)/T \to m_1$ when $T \to \infty$. Indeed, $f^T(y) = T * m_1$ when T is even, and $f^T(y) = (T + 1)/2 * \rho + (3T + 1)/2 * \tau$ when T is odd.

4. The graph in Fig.1(4) consists of 25 nodes, one for each intersection. The figure emphasizes only B_1. For this graph, $m_1 = 5\rho + 6\tau$, $m_2 = (7/3)\rho + (10/3)\tau$, $m_3 = (4/3)\rho + (7/3)\tau$, $m_4 = (3/5)\rho + (8/5)\tau$, $m_5 = (1/2)\rho + (3/2)\tau$, and $m6 = \tau$. In this case $m = m_1 = max\{m_i\}$.

5. Consider again the graph in Fig.1(4), but this time with five base nodes consisting of the whole fourth row (from, say, top to bottom). The sphere S_1 consists then of ten nodes, namely rows three and five. In this case $max\{m_i\} < m$ since one cannot take advantage of all ten nodes to balance the traffic load.

7 Achievable lower bounds

In this section we discuss the question of whether the lower bounds of Theorem 1 can be achieved, and give some positive results for square, nxn-networks (Fig. 1 (4) depicts a 5x5-network). To make the question more precise, we formulate the following conjectures about sensor networks satisfying the conditions in Section 2:

Conjecture 1. $max\{m_i\} = m_1$. A stronger form of this conjecture is: $m_1 \geq m_1 ... \geq m_n$.

Conjecture 2. With a judicious choice of base nodes it is possible to realize m_1, i.e. find a routing whose energy consumption equals $T * m_1$ (perhaps on average , as in Example 3 above).

Note that for a sensor network that satisfies both conjectures, $m^T = T * m_1$. This follows from the following chain of (in)equalities, where y_0 denotes a routing as in Conjecture 2:

$$T * m_1 = T * max\{m_i\} \leq m^T \leq f^T(y_0) = T * m_1$$

where the equalities at both extremes follow from conjectures 1 and 2, the first inequality from Theorem 1 (ii), and the second from the definition of m^T. Hence all inequalities are actually equalities, as desired.

Example 2 above shows that Conjecture 1 is not true in general. In all examples of square networks we have computed we have found, however, that $m_1 \geq .. \geq m_n$. Examples 3 and 5 show that Conjecture 2 is false even for square networks. For certain square networks we have, however, the following positive result:

Theorem 3. *Suppose given a square, nxn-network, with exactly one base node. Consider the following two possible locations for the base node: a) at the lower left-hand corner, and b) at the center of the square (when n is even, the center consists of a central square with four nodes; choose, for definitiveness, the lower lefthand side corner). In both cases, Conjecture 1 holds (in fact, in its strong form). Also Conjecture 2 holds in both cases (exactly, for n odd, and on average, when n is even).*

Proof. We give a proof in case a), leaving the proof of b) to the interested reader. We use integer coordinates (j, k) for $j, k = 1, 2, .., n$ to denote the nodes of the network, and let $N = n^2$ denote the total number of nodes. With the notation of Section 4, S_i, the sphere of radius i, consists of nodes (j, k) such that $j + k = i + 2$. Geometrically, the spheres can be pictured as segments parallel to the main anti-diagonal of the square which, by the way, is exactly S_{n-1}. It is easy to see that $s_i = i + 1$ for $i = 1, ..., n - 1$, and

$$s_i = s_{2(n-1)-i} \tag{5}$$

for $i = n, ..., 2(n - 1)$ (geometrically, this corresponds to flipping the square along the main diagonal). Observe that, for $i \leq n-1, b_i = 1+2+...+i+1 = (i+1)*(i+2)/2$. On the other hand,

$$N - b_j = b_{2(n-1)-j-1} \tag{6}$$

for $j = n - 1, n, ..., 2(n - 1)$. This follows from the fact that $N - b_j =\mid V - B_j \mid$, the number of nodes in the complement of B_j , the ball of radius j. On the other hand, $V - Bj = S_{j+1} \cup ... \cup S_{2(n-1)}$. Hence $N - b_j = s_{j+1} + ... + s_{2(n-1)}$ and, using (5), we get $N - b_j = s_{2(n-1)-j-1} + ... + s_1 + s_0 = b_{2(n-1)-j-1}$, as desired.

We can now prove Conjecture 1. Notice that $m_i \geq m_{i+1}$ will follow from $(N - b_i)/s_i \geq (N - b_{i+1})/s_{i+1}$. Since by definition $b_{i+1} = s_{i+1} + b_i$, this last inequality is equivalent to

$$s_i * s_{i+1} \geq (N - b_i) * (s_i - s_{i+1}) \tag{7}$$

for $i = 1, ..., 2(n-1)-1$. In this range, both the left-hand side of (7) and $N - b_i$ are positive. Thus, (7) holds trivially for $i \leq n-2$, because in this range $s_i - s_{i+1} = -1$. Suppose now that $i \geq n-1$. Then (7) reduces to $s_i * s_{i+1} \geq (N - b_i)$ or, using (6), to $s_i * s_{i+1} \geq b_{2(n-1)-i-1}$. We complete the proof by showing that the left-hand side is twice as large as the right-hand side. Indeed, notice that the assumption $i \geq n-1$ implies $2(n-1)-i-1 \leq n-2$ and hence $b_{2(n-1)-i-1} = [2(n-1)-i] * [2(n-1)-i+1]/2$. On the other hand, it follows from (5) that $s_i = s_{2(n-1)-i} = 2(n-1)-i+1$, and $s_{i+i} = s_{2(n-1)-i-1} = 2(n-1)-i$, as desired. This proves Conjecture 1.

To prove Conjecture 2 we construct X_1 and X_2, two trees in the network rooted, respectively, at (1,2) and (2,1) (see Fig. 2). X_1 consists of the vertical segment of points with real coordinates $(1, y)$, for $2 \leq y \leq n$, together with a number of horizontal segments, as follows. Two segments with points of the form $(x, 3)$ and $(x, 4)$ with $1 \leq x \leq 3$, two more segments with points of the form $(x, 5)$ and $(x, 6)$ with $1 \leq x \leq 5$, etc. The construction stops when the second coordinate of the segments reaches n. Clearly X_1 is a tree rooted at $(1, 2)$. Similarly, X_2 consists of the horizontal segment $(x, 1)$ with $2 \leq x \leq n$, and a number of vertical segments, as follows. Two segments with points $(2, y)$ and $(3, y)$ with $1 \leq y \leq 2$, two more segments of the form $(4, y)$ and $(5, y)$ with $1 \leq y \leq 4$, etc. As before, X_2 is a tree, rooted at $(2, 1)$. By construction, the difference between the number of nodes in X_1 minus the number of nodes in X_2 is zero, when n is odd, and 1, when n is even.

To prove Conjecture 2 when n is odd, let y_1 denote the routing (with no loops) defined by the two trees, and set $y = (y_1, y_1, ...)$ (y has T coordinates). y balances the traffic load exactly, so that at every iteration the exact same number of packets reaches the base through $(1, 2)$ as it does through $(2, 1)$. Thus $f^T(y) = T * m_1$, as desired. When n is even, we need two new trees X^1 and X^2 obtained, respectively, by flipping X_1 and X_2 along the main diagonal. Notice that they switch roots but have the same number of nodes. For instance, X^1 is rooted at $(2, 1)$ but has the same number of nodes as X_1. Let y_1 denote the routing defined by X_1 and X_2, and y_2 the routing defined by X^1 and X^2. Given a number T of iterations, we set $y = (y_1, y_2, y_1, y_2, ...)$ (note that y has T coordinates). When T is even, y balances the traffic load exactly, so that after T iterations the same number of packets will have reached the base through $(1, 2)$ and $(2, 1)$. When T is odd there will be a difference of a packet in the traffic that passes through these two nodes. In this case, Conjecture 2 holds on average, as desired. This completes the proof of the theorem.

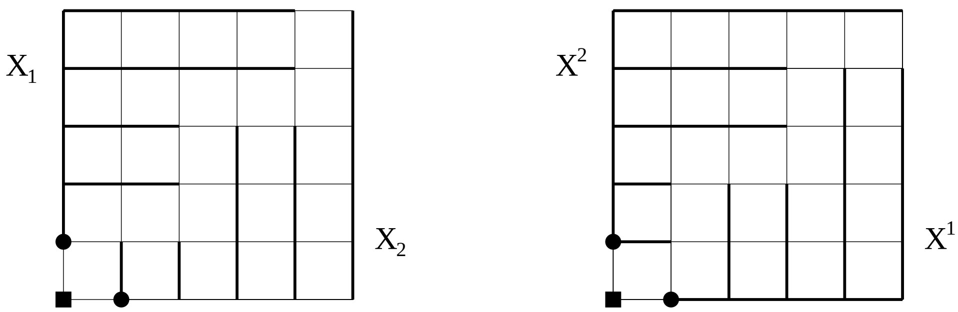

Figure 2: Routing trees in a 6x6-network (X_1 contains 17 nodes and X_2 18)

8 An application to square-lattice sensor networks

Given a square region of side L, we consider a family $\{SN_x | \sqrt{x} = 2, 3, ..\}$ of sensor networks deployed in the region. SN_x consists of $x = \sqrt{x} \times \sqrt{x}$ sensor nodes that form a square lattice consisting of squares of side $L/\sqrt{x}$ (we call these small squares the *building blocks* of SN_x). Figures 1(4) and 2 show 5x5 and 6x6-networks. More explicitly, suppose the corners of the region have coordinates $(0, 0)$, $(L, 0)$, $(0, L)$ and (L, L). Then the sensor nodes of SN_x will have coordinates ((1/2 + i)L/$\sqrt{x}$, (1/2 + j)L/ $\sqrt{x}$), for all $i, j = 0, 1, ..., \sqrt{x} - 1$ (we are assuming, of course, that $\sqrt{x}$ is an integer). Every SN_x has a chosen set B_x of base node(s). Even though B_x depends on x, we will assume that $|B_x|$, the number of base nodes, is constant and, similarly, that the number of elements in S_1 , the sphere of radius 1 around B_x , is also constant. As an example, consider the case when B_x consists of a corner node, as in Fig. 2. For different values of x the coordinates of this corner node and of the corresponding nodes in the sphere of radius one will change, but in all cases $|B_x| = 1$ and the sphere S_1 has exactly 2 elements. Since the node density is directly proportional to x, as x increases the SN_x will cover the region more densely. We call SN_x *sparse* if the minimal distance d between nodes is "large", more precisely, if $d = L/\sqrt{x} > d_0$.

In this section we study the energy consumption of the SN_x assuming that these networks satisfy the conditions of Section 2. We show in Theorem 6 that when conjectures 1 and 2 are satisfied, there is a sensor network (denoted SN_{x0}) among all the SN_x that minimizes energy consumption, and that SN_{x0} is sparse. The intuition behind this result is as follows. Since L is fixed, the SN_x become increasingly dense as x increases, thus decreasing the minimal distance between nodes in SN_x. When x is small this distance is large, and the transmission energy cost is high. This cost will decrease with increasing x. At the other extreme, when x is very large, the minimal distance between nodes is small, but there are so many sensor nodes that the traffic volume dominates over the transmission cost, making the energy needed to operate the network very large. Theorem 6 proves that there is an equilibrium point where energy consumption achieves a global minimum and shows (together with Theorem 5) how to calculate this point. We also show (see Remark 2 below) the rather curious fact that the building blocks of SN_{x0} have an essentially constant side length *regardless* of the size of L (provided that L is not too small).

Sparse sensor networks have been studied, in another context, in [6]. It should be noticed, however, that "sparse" for us is formally defined by the condition $L/\sqrt{x} > d_0$, while its use in [6] is more informal. We use the following radio energy model, a slight generalization of the one used in [3, 1]. The energy consumed by the reception of one bit of data is constant,$\rho = p$, and the energy cost to transmit one bit of data is given by

$$\tau = \tau(d) = q + k_l * d^l \tag{8}$$

where q, k_l are constants, d is the distance reached by the transmission at the given power, and l is either 2 or 4. There is a break distance $d_0 > 0$ having the property that $\tau(d) = q + k_2 * d^2$ for $d \leq d_0$, and $\tau(d) = q + k_4 * d^4$ for all distances $d > d_0$. The values used are p = 45 nJ, q = 135 nJ, $k_2 = 10^{-2}$, $k_4 = 10^{-6}$, and d_0 = 87 meters.

Using (8) we can express the value of m_1 (see (2)) for the network SN_x , as a function of x, as follows:

$$m_1(SN_x) = m_1(x) = \frac{x - b_1}{s_1} * p + \frac{x - b_1 + s_1}{s_1} * (q + k_l * \frac{L^l}{x^{l/2}}) \tag{9}$$

Let $m^T(SN_x)$ denote the value m^T (see Section 4) for the network SN_x. The following result follows immediately from Theorem 1 (ii).

Theorem 4. *Suppose given L and SN_x as above. Then for all x such that $\sqrt{x} = 2, 3, ...,$*

$$T * m_1(SN_x) \leq m^T(SN_x).$$

We now study $m_1(x)$ in itself, as a function of positive real numbers (so $\sqrt{x}$ is not necessarily an integer anymore). Theorem 5 says, roughly speaking, that in an appropriate interval, $m_1(x)$ is a U-shaped curve with a unique global minimum at a point x_0 in the interior of the interval, see Fig. 3. The jump occurs at $L/\sqrt{d_0}$ and reflects the two definitions for $m_1(x)$ (i.e. the two values of l in Equation 8).

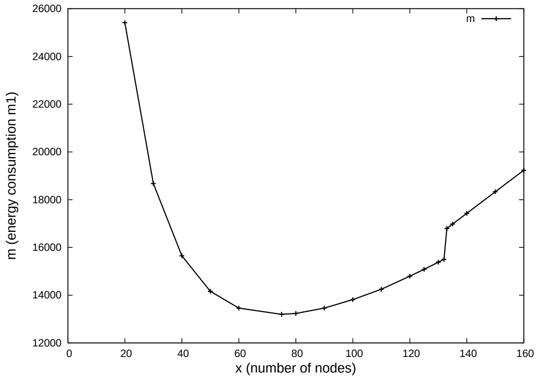

Figure 3: The function m(x) for L = 1000

We apply this rather technical result to square-lattice sensor networks in Theorem 6 below. To simplify the notation, we write $m(x) = m_1(x)$, $b = b_1$, and $s = s_1$ in the theorem below.

Theorem 5. Suppose that the following conditions hold:

i) $p + q \geq k_4 * d_0^4$

ii) $32 * (p + q) * (b - s)^2 < k_4 * L^4$

Then $m(x)$ is a convex function in the interval $[4 * (b - s), L^2/d_0^2]$ and it has a unique minimum at an interior point x_0 of this interval. If, moreover,

iii) $d_0^2 * k_4 \leq k_2$

then $m(x)$ is increasing in $[x_0, \infty)$. Consequently, x_0 is the unique global minimum of $m(x)$ in the interval $[4 * (b - s), \infty)$.

Proof. We assume i) and ii) and prove that $m(x)$ is convex in $[4(b - s), L^2/d_0^2]$. Notice that in this interval $l = 4$, and we can write $m(x) = (1/s) * [p(x - b) + (x + s - b)(q + k_4 * L^2/x^2)]$. This function has derivative $m'(x) = (1/sx^3)[2 * b * k_4 * L^4 + (p + q)x^3 - k_4 * L^4 * (2s + x)]$, and second derivative $m''(x) = (1/s * x^4) * [2 * k_4 * L^4 + (x + 3(s - b))]$. Clearly, $m''(x) > 0$ if $x > 4(b - s)$, showing that $m(x)$ is convex, and that m$'(x)$ is increasing. To prove the existence and uniqueness of x_0 , we show that $m'(x)$ takes opposite signs at the endpoints of the interval. This will show $m'(x)$ has a unique zero x_0 inside the interval, as desired.

Now $m'(L2/d_0^2) > 0$ if and only if $L^2[2 * b * k_4 + L^2 * (p + q)/d_0^6 - k_4 * (L^2/d_0^2 + 2s)] > 0$. This inequality is, in its turn, equivalent to $(L^2/d_0^2) * (p + q - k_4 * d_0^4) > s * k_4 * d_0^4 * (s - b)$, which is true because the left hand side is ≥ 0 by i), while the right hand side is strictly negative because $s - b < 0$ (notice that b - s is the number

$|B_x|$ of base nodes). Thus, $m'(L^2/d_0^2) > 0$. On the other hand, $m'(4*(b-s)) < 0$ if and only if $64*(p+q)*(b-s)^3 < 2k_4 * L^4 * (b-s)$, and this is equivalent to ii). Thus $m'(4*(b-s)) < 0$, as desired.

Suppose now that iii) holds. Let $m_{l=4}(x)[m_{l=2}(x), \text{resp.}]$ denote Equation 9 with $l = 4[l = 2, resp]$. We will show that iii) implies $m_{l=4}(L^2/d_0^2) \leq m_{l=2}(L^2/d_0^2)$. Since

$$m_{l=2}(x) - m_{l=4}(x) = \frac{x-b-s}{s} * \frac{L^2}{x} * (k_2 - k_4 * \frac{L^2}{x})$$

we get $m_{l=2}(L^2/d_0^2) - m_{l=4}(L^2/d_0^2) = (d_0^2/s) * ((L^2/d_0^2) - (b-s)) * (k_2 - k_4 * d_0^2) \geq 0$ if and only if $L^2/d_0^2 \geq b-s$, since the last factor above is ≥ 0 by iii). We now claim that $L^2/d_0^2 \geq b-s$ follows from i) and ii). Indeed, $k_4 * L^4 > 32(p+q)(b-s)^2 \geq 32 * k_4 * d_0^4(b-s)$, and this implies $L^2/d_0^2 \geq \sqrt{32}*(b-s) > b-s$, as desired.

Next, we show that $m_{l=2}(x)$ is increasing in the interval $[L2/d_0^2, \infty)$. This follows immediately from the fact that its derivative $m'_{l=2}(x) = (1/s*x^2)(k_2 * L^2 * (b-s) + (p+q)x^2) > 0$ for all $x \neq 0$. All these facts together show that $m(x)$ is increasing in the interval $[4(b-s), \infty)$ and, consequently, that $m(x)$ has a global minimum at x_0. This completes the proof of the theorem.

Remark 1. In a concrete case, when the actual values of the parameters p, q, b, s, k_4 and L are known, the value of x_0 is computed by solving the third degree equation $2bk_4L^4 + (p+q)*x^3 - k_4L^4*(2s+x) = 0$, since the roots of this equation are exactly the roots of $m'(x) = 0$.

Corollary 3. *Suppose that conditions i) and ii) of Theorem 5 hold. Then*

$$(L^4/x_0^2) = \frac{p+q}{k_4 * (1 - \frac{2*(b-s)}{x_0})} \tag{10}$$

Moreover, for all $x \in [4(b-s), \infty)$,

$$L^4/x_0^2 \in (\frac{p+q}{k_4}, 2 * \frac{p+q}{k_4}] \tag{11}$$

Proof. To prove (10), notice that x_0 satisfies $2b*k_4*L^4 + (p+q)x_0^3 = k_4L^4(2s + x_0)$, or $(p+q)x_0^3 = k_4*L^4*(x_0+2(s-b))$. Equivalently, $(p+q)*x_0 = k_4(L^4/x_0^2)*(x_0+2(s-b))$, and (10) follows. (11) follows from the fact that the right-hand side of (10) is a decreasing function of x_0 , with a flat tail. Since by assumption $x_0 > 4(b-s)$, we see that $L^4/x_0^2 < 2(p+q)/k_4$ for all L (as long as they satisfy condition i) and ii), of course). On the other hand, for large values of x_0,

$$L^4/x_0^2 \approx \frac{p+q}{k_4} = \lim_{x_0 \to \infty} \frac{p+q}{k_4 * (1 - \frac{2(b-s)}{x_0})} \tag{12}$$

Hence, $L^4/x_0^2 \in ((p+q)/k_4, 2(p+q)/k_4)$. The proof is complete.

Theorem 6. *Let conditions i)-iii) of Theorem 5, as well as conjectures 1 and 2 hold. Then the most efficient* SN_x *to cover a square region of side L is either* $SN_{\lfloor x_0 \rfloor}$ *or* $SN_{\lfloor x_0 \rfloor + 1}$. *Both networks are sparse, and the minimal energy consumed after T iterations is given by:*

$$min\{m^T(SN_x) | \sqrt{x} = 2, 3, ...\} = T * min\{m((\lfloor \sqrt{x_0} \rfloor)^2 \rfloor), m((\lfloor \sqrt{x_0} \rfloor + 1)^2)\} \tag{13}$$

where x_0 is as in Theorem 5.

$\lfloor x \rfloor$ denotes the largest integer j satisfying $j \leq x$. "Most efficient" means that the energy consumed by the network is optimal, i.e. less than or equal to the energy consumed by any other SN_x. The following corollary follows immediately from Theorem 6 (see Section 5).

Corollary 4. *The maximum number T_{max} of readings a sensor network SN_x can take under the assumptions given in Theorem 6, is bounded as follows:*

$$T_{max} \leq \frac{EE}{min\{m((\lfloor\sqrt{x_0}\rfloor)^2), m((\lfloor\sqrt{x_0}\rfloor + 1)^2)\}}$$

Remark 2. Corollary 3 shows that the larger x_0 is, the closer L^4/x_0^2 is to $(p+q)/k_4$. It follows that the L^4/x_0^2 is essentially constant (and very close to $(p+q)/k_4$) for large values of L, since x_0 grows monotonically with L. Since $\lfloor x_0 \rfloor$ is close to x_0, the same conclusion applies to $d^4 = L^4/(\lfloor\sqrt{x_0}\rfloor)^2$ and, even more so, to d, the minimal distance between the sensors in $SN_{\lfloor x_0 \rfloor}$.

Proof. Since both conjectures hold, $m^T(SN_x) = T * m(x)$ (see Section 7) for all x such that $\sqrt{x} = 2, 3, ..$ Hence, $min\{m^T(SN_x)|\sqrt{x} = 2, 3, ...\} = T * min\{m(x)|\sqrt{x} = 2, 3, ...\}$. By Theorem 5, $m(x)$ is decreasing in $[4(b-a), x_0]$, hence $min\{m(x)|\sqrt{x} = 2, 3, ... \wedge x \leq x_0\} = m((\lfloor\sqrt{x_0}\rfloor)^2)$. Similarly, since $m(x)$ is increasing in $[x_0, \infty)$, $min\{m(x)|\sqrt{x} = 2, 3, ..., \wedge x \geq x_0\} = m((\lfloor\sqrt{x_0}\rfloor + 1)^2)$. (13) follows immediately from this.

Example 6. Take L = 2000, and B_x to be the lower-left hand side corner (so that $b = 3$ and $s = 2$). Then $x_0 = 297.137$, and $d = L/\sqrt{x_0} = 116.025$. The most efficient SN_x is SN_{17} , with 289 $= (\lfloor\sqrt{x_0}\rfloor)^2 =$ 17x17 sensor nodes, and minimal distance $d =$ 117.6 meters. On the other hand $d \approx= (p + q)/k_4 =$ 115.8 m. The minimum energy consumption is $m(x_0) =$ 53440.3 nJ. We can also see that a 52.5% increase in density, from 289 to $441 =$ 21x21 sensors, will result in a 7.8% increase in the energy consumption, from $m(289) =$ 53460.9 to $m(441) =$ 57654.5. Had we instead taken B_x so that $b =$ 10 and $s =$ 8, then $x_0 =$ 296.122 and $d = L/\sqrt{x_0} = 116.224$. The most efficient SN_x is again SN_{17} , but the minimum energy consumption is now $m(x_0) =$ 13281.1. As before, an increase from 289 to 441 sensors results in 7.99% increase in the energy consumption, from $m(289) =$ 13285.0 to $m(441) = 14347.1$.

9 Simulation Experiments

The mathematical model presented in the previous sections assumes an idealized communication model, with perfect transmission scheduling and without packet losses. In order to try out the result in a more realistic scenario we have simulated the square lattice networks with varying packet loss rates. The simulations were done with the OMNet++ discrete event simulator [7]. In the simulations, we use the constant values described above and vary the length of the side L, and the number of nodes. Each sensor node periodically reads its sensor value and sends it towards the base station which is located in a top left corner of the lattice network. Packets are routed using the routes constructed in Section 7.

9.1 Including Packet Loss

In the next experiments, we use simulation results to study the impact of packet loss and retransmissions. We use a collision free MAC layer with link layer acknowledgments.

L	packet loss rate	number of retransmissions	(increase of) arrived packets	(increase of) energy consumption	X_0
600	5%	0	2692 (76.9%)	3960800	6
600	5%	1	+21.1%	+ 52.5%	5
600	5%	2	+1.7%	+4.2%	5
600	10%	0	2053 (58.6%)	3270800	6
600	10%	1	+28.3%	+78.5%	5
600	10%	2	+9.3%	+5.8%	5
600	20%	0	1250 (35.7%)	2269950	6
600	20%	1	+41.9%	+264%	5
600	20%	2	14.7%	+25.9%	5

Table 1: Energy consumption and arrived packets as a function of L, loss rate and number of retransmissions

An acknowledgment packet is $1/5$ of the size of a data packet. Therefore, in our simulations the loss probability is lower for acknowledgment packets than for data packets. Packet loss is detected by the absence of an acknowledgment packet, and the number of retransmissions is varied between the simulations. Every sensor performs 100 sensor readings and is thus the source of 100 packets.

Table 1 shows the results for $L = 600$, with packet loss rates of 5, 10 and 20% and with zero to two retransmissions. In the rows with zero retransmissions, we report absolute values of the energy consumption, the absolute number of packets that arrived at the base and the percentage of the packets that arrived at the base in paranthesis. The rows with two retransmissions present the additional difference when two retransmissions compared with only one are allowed. For example, the row with $L = 600$, packet loss rate 5% and two retransmissions shows that by allowing two retransmissions we increase the number of arrived packets by 1.7% while increasing the energy consumption with 5.1% compared to allowing only one retransmission. The results in the table indicate that performing one retransmission siginificantly increases the number of arrived packets, but that the cost in terms of increased energy consumption can be substantial. Allowing two retransmissions leads to a larger number of arriving packets but costs additionally. Of course, choosing an appropriate number of retransmissions depends on the requirements of the application.

For a given L, let x_0 be as in Theorem 5, and define z_0 to be either $\lfloor\sqrt{x_0}\rfloor$ or $\lfloor\sqrt{x_0}\rfloor + 1$, depending on which one has the least m-value. This simply means that $SN_{z_0^2}$ is the most efficient SN_x, in the sense of Theorem 6. It is interesting to note that introducing packet loss z_0 changes. In the theoretical model of Theorem 6, the optimal point z_0 is 5 for $L = 600$. When we do not perform retransmissions, z_0 increases to 6, but is 5 when retransmissions are performed. We have observed this behavior also for other values of L. When z_0 increases, the maximum number of hops the packets from remote sensors must perform also increases. By remote we mean sensors at a large distance from the base. The larger the number of hops and the higher the error rate, the more energy can be saved by dropping packets from remote sensors after only a few hops. This pushes the minimum energy consumption for higher error rates to constellations with more hops, i.e. with larger z_0 values. For L = 900 and a packet loss rate of 20%, z_0 is 10 when no retransmissions are performed while z_0 is 8 both in the theoretical case of Theorem 6, and when we perform retransmissions.

10 Conclusions and future work

In this paper, using an idealized mathematical model, we have quantified the fundamental role played by the spheres of different radii in determining the energy consumption of routings in networks satisfying the assumptions of Section 2. We have computed the theoretical optimal value, and applied this to bound the lifetime of sensor networks. We have given some examples to illustrate the theory we developed, and we have identified some general situations when the bounds are achieved, thus giving us the optimal energy consumption that can be achieved by routings.We have applied the theory to the problem of covering a given square region with square-lattice sensor networks of increasing density. We have shown that there is an essentially unique member of this family that consumes least energy, and computed its size. Finally, we have used simulation to test our results in a more realistic scenario, where packet loss occurs. For future work, we plan to test our results in a real world test bed consisting of sensor nodes equipped with radio transmitters with variable energy consumption.

11 Acknowledgements

This work was financed by VINNOVA, the Swedish Agency for Innovation Systems.

References

[1] M. Bhardwaj and A. Chandrakasan. Bounding the lifetime of sensor networks via optimal role assignments. In *IEEE Infocom*, New York, NY, USA, June 2002.

[2] J. Chang and L. Tassiulas. Energy conservin routing in wireless ad-hoc networks. In *IEEE Infocom*, pages 22–31, 2000.

[3] W. Heinzelman, A. Chandrakasan, and H. Balakrishnan. An application-specific protocol architecture for wireless microsensor networks. *IEEE Transactions on Wireless Communications*, 1(4), 2002.

[4] A. Mainwaring, J. Polastre, R. Szewczyk, D. Culler, and J. Anderson. Wireless sensor networks for habitat monitoring. In *First ACM Workshop on Wireless Sensor Networks and Applications (WSNA 20029*, Atlanta, GA, USA, September 2002.

[5] V. Raghunathan, C. Schurgers, S. Park, and M. Srivastava. Energy aware wireless microsensor networks. *IEEE Signal Processing Magazine*, 19(2):40–50, March 2002.

[6] R. Shah, S. Roy, S. Jain, and W. Brunette. Data mules: Modeling a three-tier architecture for sparse sensor networks. In *IEEE Workshop on Sensor Network Protocols and Applications*, May 2003.

[7] A. Varga. The omnet++ discrete event simulation system. In *European Simulation Multiconference*, Prague, Czech Republic, June 2001.

[8] Y. Xu, J. Heidemann, and D. Estrin. Adaptive energy-conserving routing for multihop ad hoc networks. Technical Report 527, USC/Information Sciences Institute, 2000.